Cor 2.8 (Jordan–Hölder Theorem) Any two composition series of a module M_R are equivalent.

Proof: Let $0 = A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_m = M$, $0 = B_0 \subsetneq B_1 \subsetneq \ldots \subsetneq B_n = M$ be composition series. By Thm 2.7 they have equivalent refinements $\{A_{ij}\}$, $\{B_{ji}\}$. Some factors A_{ij}/A_{ij-1}, B_{ji}/B_{ji-1} may be zero, but the nonzero ones correspond to the composition factors of the respective series. The nonzero factors must be paired with the nonzero factors in the equivalence, and the claim follows. $\square$

Def: Let $M \in \text{Mod-}R$. The length of M, $\ell(M) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is the length of a composition series if one exists, $\ell(M) = \infty$ otherwise. M has finite length if $\ell(M) < \infty$.

Lemma 2.9 Let $M \in \text{Mod-}R$. TFAE:

(a) M has a composition series

(b) $\ell(M) < \infty$

(c) M is noetherian and artinian.

Proof: (a) $\Leftrightarrow$ (b) by definition.

(a) $\Rightarrow$ (c) Let $0 = A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_n = M$ be a composition series. If $B_1 \subsetneq B_2 \subsetneq \ldots \subsetneq B_{m-1}$ is any chain of submodules, then $m \leq n$ by Thm 2.7. In particular, there are no infinite ascending or descending chains.

(c) $\Rightarrow$ (a) Recursive definition of A_i: $A_0 := 0$.

Suppose we have $A_0 \subsetneq A_1 \subsetneq \dots \subsetneq A_{i-1}$ s.t. A_j / A_{j-1} is simple $\forall 1 \le j \le i-1$.

If $A_{i-1} = M$, this is a composition series, we are done.

If $A_{i-1} \subsetneq M$, the set $\Omega = \{A \le M_R : A_{i-1} \subsetneq A_i\}$ has a minimal element A_i (by artinianity) – so A_i / A_{i-1} is simple.

This process stops after finitely many steps by noetherianity. $\square$

2.3 Semisimple Modules

Recall: If $M \in \text{Mod-}R$, and $(M_i)_{i \in I}$ is a family of submodules, then $\sum_{i \in I}' M_i$ is the smallest submodule of M containing all M_i.

Elements: $\sum_{i \in I}' m_i$ with $m_i \in M_i$, only finitely many m_i nonzero.

$\sum_{i \in I}' M_i$ is direct (an internal direct sum) if

$$\forall i \in I: \quad M_i \cap \sum_{j \in I \setminus \{i\}}' M_j = 0.$$

Then $\sum_{i \in I}' M_i \cong \bigoplus_{i \in I} M_i$ (external direct sum)

Def: $M \in \text{Mod-}R$ is semisimple (= completely reducible) if it is a direct sum of simple modules.

Exm: ·) simple modules are semisimple

·) If D is a division ring, each D-module V has a basis $(e_i)_{i \in I}$, i.e., $V = \sum_{i \in I}' e_i D$ (direct), with $e_i D$ simple, so every module is semisimple.

- 0 is semisimple, not simple
- $\mathbb{Z}_{\mathbb{Z}}$ is not semisimple, $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ (p prime) is semisimple, $\mathbb{Z}/p^2\mathbb{Z}$ not.
- D division ring, $M_n(D) = S_1 \oplus \_ \oplus S_n$ with $S_i = E_{ii} M_n(D)$ .. matrices where all entries outside the i-th row are zero.

  S_i is simple as right $M_n(D)$-module, so $M_n(D)_{M_n(D)}$ is semisimple.

<u>Lemma 2.10</u> If $M = \sum_{i \in I}' M_i$ with <u>simple</u> $M_i \leq M$, and $N \leq M$, then exists $I' \subseteq I$ s.t. $M = N \oplus \bigoplus_{i \in I'} M_i$.

<u>Proof:</u> Let $\Omega := \{ J \subseteq I \; : \; N + \sum_{j \in J}' M_j \text{ is direct} \}$

Then $\Omega \neq \emptyset$ since $\emptyset \in \Omega$. If $\Omega' \subseteq \Omega$ is a chain w.r.t. $\subseteq$, then $J' := \bigcup_{J \in \Omega'} J \in \Omega$ [If not, there exists $n + \sum_{j \in J'}' m_j = 0$, not all $m_j = 0$, but only finitely many nonzero. So this sum is actually supported on some $J \in \Omega'$ $\lightning$.]

Zorn's lemma $\Rightarrow$ Ω has a maximal element I'.

Let $M' := \sum_{i \in I'}' M_i$.

<u>Claim</u>: $M = N + M'$.

$\forall i \in I$: $M_i \cap (N+M') \in \{0, M_i\}$ by simplicity.

But $M_i \cap (N+M') = 0$ ↯ I' maximal in Ω, so

$M_i \cap (N+M') = M_i \Rightarrow M_i \subseteq N+M' \Rightarrow M = N + \sum'_{i\in I} M_i \subseteq N+M'$! $\square$

<u>Lemma 2.11</u>: Let $0 \neq M \in$ Mod-R. Suppose that for every $N \leq M$, there exists $K \leq M$ s.t. $M = N \oplus K$. Then M contains a simple submodule.

<u>Proof</u>: The assumption also holds for all $M' \leq M$:

If $N \leq M'$, $\exists K$: $M = N \oplus K$. Then $M' = N \oplus (K \cap M')$ (<u>because $N \subseteq M'$!</u>)

Thus w.r.l. $M = mR$, $m \neq 0$. By Zorn's lemma, there exists $N \leq M$ s.t. N is maximal with $m \notin N$. By assumption, there exists K s.t. $M = N \oplus K$.

If $0 \neq K' \leq K$, then $N \oplus K' \ni m$ by maximality of N.

Then $M = N \oplus K'$, hence $K' = K$. Thus, K is simple. $\square$

<u>Thm 2.12</u> For $M \in$ Mod-R, TFAE:

(a) M is semisimple

(b) M is a sum of simple modules.

(c) For every $N \leq M$, there exists $L \leq M$ s.t. $M = L \oplus N$.

<u>Proof</u>: (a) $\Rightarrow$ (b) ✓

(b) $\Rightarrow$ (c) Let $M = \sum'_{i \in I} M_i$ with M_i simple.

$\overset{L2.10}{\Rightarrow} \exists I' \subseteq I$ s.t. $M = N \oplus \bigoplus_{i \in I'} M_i$. Take $L := \bigoplus_{i \in I'} M_i$

(c) $\Rightarrow$ (a) Let N be the sum of all simple submodules of M.

$\Rightarrow M = N \oplus L$ for some $L \leq M$. L also satisfies (c) but cannot contain a simple submodule. Thus $L = \underline{0}$ by L2.11. $\square$

Cor 2.13: Quotients and submodules of semisimple modules are semisimple.

Proof. For quotients, use 2.12(b) (images of simple modules are simple or $\underline{0}$). For submodules, 2.12(c) [cf. proof of L2.11]. $\square$

Remark: (1) If $M = M_1 \oplus \ldots \oplus M_k$ with simple M_i, then the M_i are unique up to isomorphism & order, since

$\underline{0} \lneq M_1 \lneq M_1 \oplus M_2 \leq \ldots \leq M_1 \oplus \ldots \oplus M_k$ is a composition series with composition factors $M_1, \ldots, M_k$.

Endomorphism Rings:

Suppose $M_R \cong M_1 \oplus \ldots \oplus M_k$.

Let $\varepsilon_i : M_i \to M$ be the canonical embedding, $\pi_i : M \to M_i$ the canonical projection, so $m = \sum_{i=1}^{k}{}' \varepsilon_i(\pi_i(m)) \; \forall m \in M$.

If $\varphi \in \mathrm{End}(M_R)$, then $\varphi(m) = \sum_{i,j=1}^{k}{}' \varepsilon_i \circ \underbrace{\pi_i \circ \varphi \circ \varepsilon_j}_{=: \varphi_{ij}} \circ \pi_j (m)$

with $\varphi_{ij} : M_j \to M_i$

Then $\mathrm{End}(M_R) \longrightarrow \bigoplus_{i,j=1}^{k} \mathrm{Hom}(M_j, M_i)$, $\varphi \mapsto [\varphi_{ij}]_{i,j=1}^{k}$ $\quad (*)$

is an isomorphism of abelian groups.

If $\varphi, \psi \in \mathrm{End}(M)$, then (easy exercise): $(\psi \circ \varphi)_{ij} = \sum_{l=1}^{k} \psi_{il} \circ \varphi_{lj}$,

so considering the RHS to be formal matrices, $(*)$ is a ring isomorphism.

Prop 2.14 Let M_R be semisimple of finite length, say

$M \cong M_1^{n_1} \oplus \ldots \oplus M_k^{n_k}$ with M_i simple, $M_i \not\cong M_j$ for $i \neq j$.

Then $\mathrm{End}(M) \cong M_{n_1}(D_1) \times \ldots \times M_{n_k}(D_k)$ with $D_i := \mathrm{End}(M_i)$ division rings.

module M $\quad$ matrix ring $M_{n_i}(D_i)$

Proof: $\mathrm{End}(M) \underset{*}{\cong} \bigoplus_{i,j=1}^{k} \mathrm{Hom}(M_j^{n_j}, M_i^{n_i}) = \bigoplus_{i=1}^{k} \mathrm{End}(M_i^{n_i}) \cong$

since $\mathrm{Hom}(M_j, M_i) = 0$ for $i \neq j$. [L2.2]

$\underset{*}{\cong} \bigoplus_{i=1}^{k} M_{n_i}(\mathrm{End}(M_i))$, and $\mathrm{End}(M_i)$ is a div. ring [L2.2] $\square$

2.4 Semisimple Rings

Def: A ring R is (right) semisimple if R_R is a semisimple module.

Rem: Later: right semisimple $\Leftrightarrow$ left semisimple.

Exm: •) D div. ring is semisimple, as is $M_n(D)$

•) R_1, R_2 semisimple $\Rightarrow$ $R_1 \times R_2$ semisimple [If M is an $R_1 \times R_2$-module then $M = M_1 \oplus M_2$ with $M_i \in \mathrm{Mod}\text{-}R_i$]

•) $\mathbb{Z}$ is not semisimple,

$\mathbb{Z}/n\mathbb{Z}$ semisimple $\Leftrightarrow$ n squarefree $\Leftrightarrow$ $\mathbb{Z}/_{n}\mathbb{Z} \cong \mathbb{Z}/_{p_1}\mathbb{Z} \times \dots \times \mathbb{Z}/_{p_r}\mathbb{Z}$

with pairwise distinct primes $p_1, \dots, p_r$.

Recall: If M, N, K are R-modules,

$0 \to M \xrightarrow{f} N \xrightarrow{g} K \to 0$ is a SES $\Leftrightarrow$ $\ker f = 0$, $\operatorname{im} f = \ker g$, $\operatorname{im} g = 0$.

(1) The SES is split exact if

$\exists f' : N \to M$ s.t. $f' \circ f = id_M$

$\Leftrightarrow$ $\exists g' : K \to N$ s.t. $g \circ g' = id_K$

$\Leftrightarrow$ $\exists \varphi : N \xrightarrow{\sim} M \oplus K$ s.t.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "id"'] & N \arrow[r] \arrow[d, "\varphi"] & K \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & M \arrow[r, hook] & M \oplus K \arrow[r, two heads] & K \arrow[r] & 0
\end{tikzcd}$$

commutes

$m \mapsto (m, 0)$

$(m, k) \mapsto k$

(2) P_R is projective if every SES $0 \to M \to N \to P \to 0$ splits.

(3) I_R is injective if every SES $0 \to I \to N \to K \to 0$ splits

Thm 2.15: TFAE (for a ring R):

(a) R is right semisimple

(b) All SES in Mod-R split

(c) All $M \in$ Mod-R are semisimple

(d) All f.g. $M \in$ Mod-R are semisimple

(e) All cyclic $M \in$ Mod-R are semisimple

Proof: (b) $\Rightarrow$ (c) Let $N \leq M$. Then $0 \to N \to M \to M/N \to 0$ is split exact by (b), so $M = N \oplus K$ with $K \cong M/N$.

$\overset{T\,2.12(c)}{\Longrightarrow}$ M semisimple.

(c) $\Rightarrow$ (d) $\Rightarrow$ (e) ✓ (e) $\Rightarrow$ (a): R_R is cyclic ✓

(a) $\Rightarrow$ (c): R_R semisimple $\Rightarrow$ $R_R^{(I)}$ semisimple for all index sets I.

Every module M_R is a quotient of some free module $R_R^{(I)}$, hence semisimple [C2.13].

(c) $\Rightarrow$ (b) Let $0 \to M \xrightarrow{f} N \xrightarrow{g} K \to 0$ be exact.

N semisimple $\Rightarrow$ $N = f(M) \oplus K'$ with $K' \leq N$ [T2.12]

Define $f': N \to M$ by $\underset{\text{projection}}{N \twoheadrightarrow f(M)} \xrightarrow{f^{-1}} M$. Then $f' \circ f = id_M$. $\square$

Cor 2.16 If R_R is right semisimple, then

$R_R = M_1^{n_1} \oplus \_\_ \oplus M_k^{n_k}$ with simple, pw. nonisomorphic $M_i \in$ Mod-R.

In particular, R_R has finite length and only finitely many simple modules $M_1, \dots, M_k$.

Proof: We know $R_R = \bigoplus_{i \in I} M_i$ with simple M_i.

But $R_R = 1 \cdot R_R$ is cyclic, and there is a finite $I_0 \subseteq I$ s.t.

$1 \in \bigoplus_{i \in I_0} M_i \Rightarrow R_R = \bigoplus_{i \in I_0} M_i$. $\square$

Remark: Similarly, if M_R is semisimple,

M_R f.g. $\Leftrightarrow$ $M \cong M_1 \oplus \_\_ \oplus M_n$, M_i simple $\Leftrightarrow$ $\ell(M) < \infty$.

Cor 2.17 TFAE:

(a) R is right semisimple

(b) Every $M \in \text{Mod-}R$ is projective.

(c) Every $M \in \text{Mod-}R$ is injective.

Proof. (a)$\Leftrightarrow$(b) and (a)$\Leftrightarrow$(c) both follow from T2.15(b) $\square$

If D_i div. ring, then $M_{n_i}(D_i)$ is semisimple. Finite products of semisimple rings are semisimple, so $M_{n_1}(D_1) \times \ldots \times M_{n_k}(D_k)$ is semisimple

Thm 2.18 (Wedderburn-Artin) If R is right semisimple, then $R \cong M_{n_1}(D_1) \times \ldots \times M_{n_k}(D_k)$ with D_i division rings, $n_i \geq 1$. k and $(D_1, n_1), \ldots, (D_k, n_k)$ are uniquely determined (up to order and isomorphism) and R has exactly k simple modules $S_1, \ldots, S_k$ up to isomorphism. Also $D_i \cong \text{End}(S_i)$.